Preliminary Exam: Electricity and Magnetism, Thursday, January 17, 2019, 9am-noon

Answer a total **THREE** questions out of **FOUR**. If you turn in excess solutions, the ones to be graded will be picked at random.

Each answer must be presented **separately** in an answer book, or on consecutively numbered sheets of paper stapled together. Make sure you clearly indicate who you are and the problem you are solving. Double-check that you include everything you want graded, and nothing else.

1. An infinitely long cylindrical conductor of radius R carries a volume current I_V propagating along the cylinder axis z. Another steady current I_S is uniformly distributed over the outside surface of the conductor and flows down in the opposite direction. The magnetic permeability of the conductor material and entire space is equal to the vacuum permeability μ_0.

   (a) Determine the magnetic vector potential $\vec{A}(s)$ as a function of the radius s in cylindrical coordinates in the entire space if $I_V = I_S$ and the volume current I_V is uniformly distributed over the conductor cross section. The z-axis can be considered as the reference line for the vector-potential: $\vec{A}(s = 0) = 0$.

   (b) Find the vector potential $\vec{A}(s)$ if $I_V = I_S + \delta I$, where δI is a positive constant.

   (c) From the expression for $\vec{A}(s)$ derived in (b), determine the leading term of the asymptotic behavior of the vector potential at $s \gg R$. Explain the fact that $\vec{A}(s)$ is divergent at large s.

   (d) Determine the force $\vec{F}$ exerted by the volume current I_V per unit area of the conductor surface carrying the skin current I_S.

2. Two non-uniformly charged disks of radius R are placed a distance d apart. The disk planes are perpendicular to the z-axis passing through the disk centers. The surface charge densities of the disks are $\sigma_+ = \sigma_0\,(s/R)^2$ on the upper disc and $\sigma_- = -\sigma_0\,(s/R)^2$ on the lower disc, where s is the distance from the disk center and σ_0 is a positive constant.

   (a) Determine the potential $\Phi(z)$ and the electric field $\vec{E}(z)$ on the disk axis.

   Hint: The following indefinite integral may be useful for your solution

$$\int \frac{t^3\,dt}{(t^2+b^2)^{1/2}} = \frac{1}{3}(t^2+b^2)^{3/2} - b^2(t^2+b^2)^{1/2} + \text{const.}$$

   (b) Find the leading term in the asymptotic expansion of the potential $\Phi(z)$ obtained in (a) at large distances $z \gg R,\ d$.

   (c) Derive the analytical expression for the potential $\Phi(\vec{r})$ at an arbitrary point of space, using the leading term of the multipole expansion for $r = \sqrt{s^2+z^2} \gg R,\ d$, where s is the radius in the cylindrical coordinate system. Does the result obtained here in part (c) match with the expression for the potential from part (b)? How does your result compare with the potential of an electric dipole $\vec{p} = Qd\vec{e}_z$ placed at the origin of the coordinate system, where Q is the total charge on the upper disk?

3. A point charge $q > 0$ is located at a distance r from the center of a conducting sphere of radius $R < r$, see Figure 1. The sphere is not grounded and carries a positive charge $Q > 0$. Find the energy of electrostatic interaction $U(r)$ between the sphere and point charge as function of distance r for $R < r < \infty$. Draw the graph of $U(r)$.

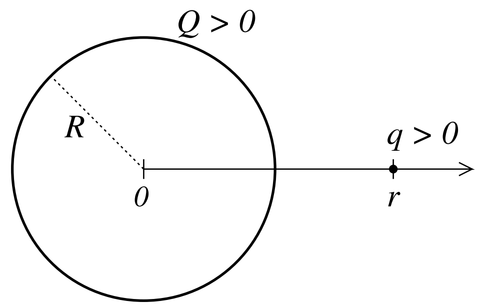

Figure 1.

4. A sphere made of a linear magnetic material with magnetic susceptibility χ_m is placed in an otherwise uniform magnetic field $\vec{B}_0$, see Figure 2. The sphere is magnetized uniformly (parallel or antiparallel to $\vec{B}_0$ depending on the sign of χ_m).

   (a) Using the fact that the magnetic field inside of a uniformly magnetized sphere with given magnetization $\vec{M}$ is given by the formula

   $$\vec{B} = \frac{2}{3}\,\mu_0\,\vec{M}\,,$$

   find the exact formula which gives the dependence of the acquired magnetization on the external field $\vec{B}_0$.

   (b) What is the total magnetic field $\vec{B}_{\text{tot}}$ inside the sphere?

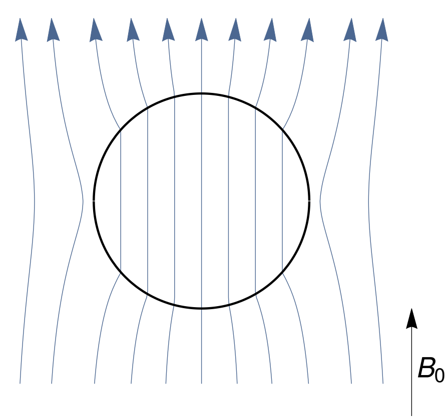

Figure 2.

Vector Formulas

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$
$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$
$$\nabla\times\nabla\psi = 0$$
$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$
$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$
$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$
$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$
$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$
$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$
$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

If $\mathbf{x}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\mathbf{x}|$, $\mathbf{n} = \mathbf{x}/r$ is a unit radial vector, and $f(r)$ is a well-behaved function of r, then

$$\nabla\cdot\mathbf{x} = 3 \qquad \nabla\times\mathbf{x} = 0$$
$$\nabla\cdot[\mathbf{n}f(r)] = \frac{2}{r}f + \frac{\partial f}{\partial r} \qquad \nabla\times[\mathbf{n}f(r)] = 0$$
$$(\mathbf{a}\cdot\nabla)\mathbf{n}f(r) = \frac{f(r)}{r}[\mathbf{a} - \mathbf{n}(\mathbf{a}\cdot\mathbf{n})] + \mathbf{n}(\mathbf{a}\cdot\mathbf{n})\frac{\partial f}{\partial r}$$
$$\nabla(\mathbf{x}\cdot\mathbf{a}) = \mathbf{a} + \mathbf{x}(\nabla\cdot\mathbf{a}) + i(\mathbf{L}\times\mathbf{a})$$

where $\mathbf{L} = \dfrac{1}{i}(\mathbf{x}\times\nabla)$ is the angular-momentum operator.

Theorems from Vector Calculus

In the following ϕ, ψ, and $\mathbf{A}$ are well-behaved scalar or vector functions, V is a three-dimensional volume with volume element d^3x, S is a closed two-dimensional surface bounding V, with area element da and unit outward normal $\mathbf{n}$ at da.

$$\int_V \nabla\cdot\mathbf{A}\,d^3x = \int_S \mathbf{A}\cdot\mathbf{n}\,da \qquad \text{(Divergence theorem)}$$
$$\int_V \nabla\psi\,d^3x = \int_S \psi\mathbf{n}\,da$$
$$\int_V \nabla\times\mathbf{A}\,d^3x = \int_S \mathbf{n}\times\mathbf{A}\,da$$
$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,d^3x = \int_S \phi\mathbf{n}\cdot\nabla\psi\,da \qquad \text{(Green's first identity)}$$
$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,d^3x = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\,da \qquad \text{(Green's theorem)}$$

In the following S is an open surface and C is the contour bounding it, with line element $d\mathbf{l}$. The normal $\mathbf{n}$ to S is defined by the right-hand-screw rule in relation to the sense of the line integral around C.

$$\int_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,da = \oint_C \mathbf{A}\cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}$$
$$\int_S \mathbf{n}\times\nabla\psi\,da = \oint_C \psi\,d\mathbf{l}$$

Explicit Forms of Vector Operations

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be orthogonal unit vectors associated with the coordinate directions specified in the headings on the left, and A_1, A_2, A_3 be the corresponding components of $\mathbf{A}$. Then

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3 \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

$$\nabla \times \mathbf{A} = \mathbf{e}_1 \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3 \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

Remark: Notice that here for cylindrical coordinates the notation (ρ, ϕ, z) is used. In the problems the notation (s, ϕ, z) is used (to avoid confusion with the charge density for which sometimes also the Greek letter "ρ" or "ϱ" is used).